# The Relation between Monotonicity and Strategy-Proofness

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# This is how it started

Olivier asked me in at the SED meeting in Bodrum 2006 (at the pool):

“For single-peaked preferences, strategy-proofness always implies Maskin monotonicity, right?”

# This is how it started

Or maybe he asked (but definitely at the pool)?

“For single-peaked preferences, Maskin monotonicity always implies strategy-proofness, right?”

# The Model

- $N = \{1, ..., n\}$ *set of agents*.
- $A = A_1 \times A_2 \times ... \times A_n$ set of *alternatives*.
- w.l.o.g., $A$ has Cartesian product form and for all $i, j \in N$, $A_i = A_j$.
- $A_i \subseteq \mathbb{R}^m$ and $|A_i| = \infty \Rightarrow A_i$ is convex.
- $F \subseteq A$ set of *feasible alternatives*.

With this general model we can easily specify

public good as well as private good and

discrete as well as continuous

economies.

# Example 1

### Example

Let $A = \{a_1, ..., a_n\} \times ... \times \{a_1, ..., a_n\}$.

Suppose that agents have to choose one candidate out of $n$ possible candidates. Then,

$$F = \{x \in A : \text{for all } i, j \in N,\ x_i = x_j\}.$$

On the other hand, if agents have to allocate $n$ indivisible tasks among themselves, then

$$F = \{x \in A : \text{for all } i, j \in N,\ x_i \neq x_j\}.$$

# Example 2

## Example

Let $A = [0,1] \times \ldots \times [0,1]$.

Suppose that the agents have to choose a single point in the interval [0,1] that everyone will consume without rivalry, e.g., a public facility on a street (Moulin, 1980). Then,

$$F = \{x \in A : \text{for all } i, j \in N,\ x_i = x_j\}.$$

On the other hand, if agents have to choose a division of a unit of an infinitely divisible good among themselves (Sprumont, 1991), then feasibility is determined by the size of the resource and

$$F = \{x \in A : \text{for all } i \in N,\ x_i \geq 0 \text{ and } \sum_{i \in N} x_i = 1\}.$$

# The Model continued

- $R_i$ denotes agent $i$'s *preferences* over $A_i$. For all $x, y \in A$,
  - $x\, R_i\, y$ is interpreted as "$i$ weakly prefers $x$ to $y$",
  - $x\, P_i\, y$ as "$i$ strictly prefers $x$ to $y$", and
  - $x\, I_i\, y$ as "$i$ is indifferent between $x$ and $y$".
- $\mathcal{R}_i = \mathcal{R}$ *set of preferences on* $A_i$ (w.l.o.g., same $\mathcal{R}$!).
- $\mathcal{R}^N$ set of *preference profiles* $R = (R_i)_{i \in N}$.
- Let $A$, $F$, and $\mathcal{R}$ be given.

  Then, a *rule* $\varphi$ is a function that assigns to every preference profile $R \in \mathcal{R}^N$ a feasible alternative $\varphi(R) \in F$.

# Strategy-Proofness and Non-Bossiness

First, we discuss the incentive property strategy-proofness, which requires that no agent ever benefits from misrepresenting his preference relation.

For agent $i \in N$, preference profile $R \in \mathcal{R}$, and preference relation $R'_i \in \mathcal{R}$, we obtain preference profile $(R'_i, R_{-i})$ by replacing $R_i$ at $R$ by $R'_i$.

## Definition (Strategy-Proofness)

A rule $\varphi$ is *strategy-proof* if for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$,

$$\varphi(R)\ R_i\ \varphi(R'_i, R_{-i}).$$

## Definition (Non-Bossiness)

A rule $\varphi$ is *non-bossy* if for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$, $\varphi_i(R) = \varphi_i(R'_i, R_{-i})$ implies that

$$\varphi(R) = \varphi(R'_i, R_{-i}).$$

# Monotonicity I

Next, we define monotonic transformations. Loosely speaking, for any alternative $x$ and any preference profile $R$, if at a preference profile $R'$ all agents $i \in N$ consider alternative $x$ to be (weakly) better, then $R'$ is a monotonic transformation of $R$ at $x$.

Let $i \in N$, $R_i \in \mathcal{R}$, and $x \in A$. Then,

- the *lower contour set of $R_i$ at $x$* is $L_i(R_i, x) \equiv \{y \in A : x_i \, R_i \, y_i\}$.
- $R'_i$ is a *monotonic transformation of $R_i$ at $x$* if $L_i(R_i, x) \subseteq L_i(R'_i, x)$.
- $MT(R_i, x)$ *set of all monotonic transformations of $R_i$ at $x$* and
- $MT(R, x)$ *set of all monotonic transformations of $R$ at $x$*, i.e., $R' \in MT(R, x)$ if for all $i \in N$, $R'_i \in MT(R_i, x)$.

# Monotonicity II

A rule $\varphi$ is Maskin monotonic if an alternative $x$ that is chosen at preference profile $R$ is also chosen at a preference profile $R'$ where $x$ is considered (weakly) better by all agents.

## Definition (Maskin Monotonicity)

A rule $\varphi$ is *(Maskin) monotonic* if for all $R, R' \in \mathcal{R}^N$,

$$\varphi(R) = x \text{ and } R' \in MT(R, x) \text{ imply } \varphi(R') = x.$$

## Definition (Unilateral Monotonicity)

A rule $\varphi$ is *unilaterally monotonic* if for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$,

$$\varphi(R) = x \text{ and } R'_i \in MT(R_i, x) \text{ imply } \varphi_i(R'_i, R_{-i}) = x_i.$$

# Rich Domains: M $\Rightarrow$ SP

Let $R_i \in \mathcal{R}$. Then, by $b(R_i)$ we denote agent $i$'s best alternatives in $A$, i.e., $b(R_i) := \{x \in A : \text{for all } y \in A,\ x_i \, R_i \, y_i\}$.

To establish our first result, we introduce the following domain "richness" condition.

## Definition (Condition R1)

Let $R_i \in \mathcal{R}$ and $x, y \in A$ such that $y \, P_i \, x$. Then, there exists $R'_i \in \mathcal{R}$ such that $y \in b(R'_i)$ and $L_i(R_i, x) \subseteq L_i(R'_i, x)$.

## Theorem

*Let $\mathcal{R}$ satisfy condition R1.*
*If $\varphi$ is unilaterally/Maskin monotonic, then $\varphi$ is strategy-proof.*

## Proof.

- **Suppose $\varphi$ is unilaterally monotonic, but not strategy-proof.**
- Then, there exist $R \in \mathcal{R}^N$, $i \in N$, and $\bar{R}_i \in \mathcal{R}$ such that

$$\varphi(\bar{R}_i, R_{-i})\, P_i\, \varphi(R).$$

- Denote $\varphi(R) = x$ and $\varphi(\bar{R}_i, R_{-i}) = y$.
- Hence, $y\, P_i\, x$ and by Condition R1 there exists $R'_i \in \mathcal{R}$ such that $y \in b(R'_i)$ and $L_i(R'_i, x) = L_i(R_i, x)$.
- Thus, $R'_i \in MT(\bar{R}_i, y)$ and $R'_i \in MT(R_i, x)$.
- By unilateral monotonicity, $\varphi_i(R'_i, R_{-i}) = y_i$ and $\varphi_i(R'_i, R_{-i}) = x_i$.
- Hence, $x_i = y_i$; contradicting our assumption that $y\, P_i\, x$.

□

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- Denote $\varphi(R) = x$ and $\varphi(\bar{R}_i, R_{-i}) = y$.
- Hence, $y\, P_i\, x$ and by Condition R1 there exists $R_i' \in \mathcal{R}$ such that $y \in b(R_i')$ and $L_i(R_i', x) = L_i(R_i, x)$.
- Thus, $R_i' \in MT(\bar{R}_i, y)$ and $R_i' \in MT(R_i, x)$.
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- **Denote $\varphi(R) = x$ and $\varphi(\bar{R}_i, R_{-i}) = y$.**
- Hence, $y \, P_i \, x$ and by Condition R1 there exists $R'_i \in \mathcal{R}$ such that $y \in b(R'_i)$ and $L_i(R'_i, x) = L_i(R_i, x)$.
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- **Hence, $y \, P_i \, x$ and by Condition R1 there exists $R_i' \in \mathcal{R}$ such that $y \in b(R_i')$ and $L_i(R_i', x) = L_i(R_i, x)$.**
- Thus, $R_i' \in MT(\bar{R}_i, y)$ and $R_i' \in MT(R_i, x)$.
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- Thus, $R_i' \in MT(\bar{R}_i, y)$ and $R_i' \in MT(R_i, x)$.
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□

# Rich Domains: SP $\not\Rightarrow$ M

## Example

We reconsider Moulin's (1980) model as introduced in Example 2. Thus, for all $i \in N, A_i = [0,1]$ and agents' preferences are single-peaked on $\mathbb{R}$.

Let $c_1, c_2 \in [0,1]$, $c_1 < c_2$, and $k \in N$. Then, for all $R \in \mathcal{R}^N$,

$$\varphi(R) := \begin{cases} c_1 \mathbb{1} & \text{if } c_1 \, P_k \, c_2 \text{ or if } c_1 \, I_k \, c_2 \text{ and } p(R_k) \in \mathbb{Q}; \\ c_2 \mathbb{1} & \text{if } c_2 \, P_k \, c_1 \text{ or if } c_1 \, I_k \, c_2 \text{ and } p(R_k) \notin \mathbb{Q}. \end{cases}$$

It is easy to see that $\varphi$ is strategy-proof, but not monotonic.

This rule can also be used to construct a private goods example.

# Restricted Domains: SP ⇒ M

For agent $i \in N$, preference relation $R_i \in \mathcal{R}_i$, and alternative $x \in A$, the *upper contour set of $R_i$ at $x$* is $U_i(R_i, x) \equiv \{y \in A : y_i \; R_i \; x_i\}$.

## Definition (Condition R2)

Let $R_i, R_i' \in \mathcal{R}$ and $x \in A$ such that $R_i' \in MT_i(R_i, x)$ and $R_i' \neq R_i$. Then, $L_i(R_i, x) \cap U_i(R_i', x) = \{x\}$.

## Theorem

*Let $\mathcal{R}$ satisfy Condition R2.*

(a) *If $\varphi$ is strategy-proof, then it is unilaterally monotonic.*

(b) *Let $F$ determine a public goods economy.*
*If $\varphi$ is strategy-proof, then it is Maskin monotonic.*

## Proof.

(b)

- **Suppose $\varphi$ is strategy-proof, but not Maskin monotonic.**
- Then, there exist $R, R' \in \mathcal{R}^N$ such that $R' \in MT(R,x)$, $\varphi(R) = x$ and $\varphi(R') = y \neq x$.
- Assume that $R' = (R'_i, R_{-i})$ for some $i \in N$.
- By strategy-proofness, $x \, R_i \, y$ and $y \, R'_i \, x$.
- Thus, $y \in L_i(R_i, x)$ and $y \in U_i(R'_i, x)$.
- Hence, $x \neq y \in L_i(R_i, x) \cap U_i(R'_i, x) = \{x\}$; a contradiction with Condition R2.
- Assume that $R' = (R'_i, R'_j, R_{-i,j})$ for some $i, j \in N$ etc.

  iteration of the argument.

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Proof.

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- Suppose $\varphi$ is strategy-proof, but not Maskin monotonic.
- **Then, there exist** $R, R' \in \mathcal{R}^N$ **such that** $R' \in MT(R,x)$**,** $\varphi(R) = x$ **and** $\varphi(R') = y \neq x$**.**
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- By strategy-proofness, $x\, R_i\, y$ and $y\, R'_i\, x$.
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- By strategy-proofness, $x \, R_i \, y$ and $y \, R'_i \, x$.
- **Thus,** $y \in L_i(R_i, x)$ **and** $y \in U_i(R'_i, x)$**.**
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- **Assume that $R' = (R'_i, R'_j, R_{-i,j})$ for some $i, j \in N$ etc. iteration of the argument.**

□

# Restricted Domains: M $\not\Rightarrow$ SP

**Example**

We reconsider Moulin's (1980) model as described in Examples 2, but with symmetrically single-peaked preferences.

Let $c_1, c_2 \in [0,1]$, $c_1 < c_2$, and $k \in N$. Then, for all $R \in \mathcal{R}^N$,

$$\varphi(R) := \begin{cases} p(R_k)\mathbb{1} & \text{if } p(R_k) \leq c_1; \\ c_2\mathbb{1} & \text{otherwise.} \end{cases}$$

It is easy to see that $\varphi$ is Maskin monotonic, but not strategy-proof.

This rule can also be used to construct a private goods example.

# Single-Peakedness on $\mathbb{R}$

*DHM domain*

*single-plateaued domain* *Cantala domain*

*single-peaked domain*

$\cup$

*symmetric single-peaked domain*

Figure: Set-relationships between single-peaked domains

# Single-Peakedness on $\mathbb{R}$

- symmetric single-peaked preferences (Euclidean) on $\mathbb{R}$

  do not satisfy R1, but they satisfy R2: SP $\overset{\not\Leftarrow}{\Rightarrow}$ M.

- single-peaked preferences on $\mathbb{R}$,
- single-peaked preferences on $\mathbb{R}$ with outside option (Cantala, 2004),
- single-plateaued preferences on $\mathbb{R}$, and
- general single-peaked preferences on $\mathbb{R}$ (Dasgupta, Hammond, and Maskin, 1979)

  satisfy R1, but they do not satisfy R2: M $\overset{\not\Leftarrow}{\Rightarrow}$ SP.

# single-peakedness on $\mathbb{R}^m$ and Arrowian preferences

- symmetric single-peaked preferences (Euclidean) on $\mathbb{R}^m$,
- separable-quadratic preferences on $\mathbb{R}^m$,
- preferences on $\mathbb{R}^m$ that are induced by a strictly convex norm,

  do not satisfy R1, but they satisfy R2: SP $\stackrel{\not\Leftarrow}{\Rightarrow}$ M.
- convex (!) star-shaped preferences on $\mathbb{R}^m$

  satisfy R1, but they do not satisfy R2: M $\stackrel{\not\Leftarrow}{\Rightarrow}$ SP.

Furthermore,

- linear orders or Arrowian preferences and
- so-called left-right (right-left) single-peaked preferences on $\mathbb{R}$

  satisfy R1 and R2: M $\Leftrightarrow$ SP.

**Theorem (An Extension of the Muller-Satterthwaite Theorem)**

*Let $\mathcal{R}$ satisfy Conditions R1 and R2 and let rule $\varphi$ be defined on $\mathcal{R}^N$.*

(a) *Then, $\varphi$ is unilaterally monotonic if and only if it is strategy-proof.*

(b) *Let $F$ determine a public goods economy. Then, $\varphi$ is Maskin monotonic if and only if it is strategy-proof.*

(c) *Then, $\varphi$ is Maskin monotonic if and only if it is strategy-proof and non-bossy.*

**Corollary (The Muller-Satterthwaite Theorem)**

*Let $A$ and $F$ be given such that $F$ determines a public goods economy. Let rule $\varphi$ be defined on the Arrovian domain $\mathcal{R}_A$. Then, $\varphi$ is Maskin monotonic if and only if it is strategy-proof.*