

Farsighted Stability for Roommate Markets

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Roommate Markets

- Gale and Shapley (1962)
- $N = \{1, \dots, n\}$: *set of agents*.
- \succsim_i : *agent i 's preferences* over sharing a room with any of the agents in $N \setminus \{i\}$ and having a room off campus.
- We assume that preferences are *strict*, e.g., $j \succ_i k \succ_i i \succ_i h \succ_i \dots$
- A *roommate market* consists of a set of agents N and their preferences \succsim and is denoted by (N, \succsim) .
- A *marriage market* is a roommate market (N, \succsim) such that N is the union of two disjoint sets M and W , and each agent in M (respectively W) prefers being single to being matched with any other agent in M (respectively W).

Blocking

- A *matching* μ for roommate market (N, \succeq) is a function $\mu : N \rightarrow N$ of order two, i.e., for all $i \in N$, $\mu(\mu(i)) = i$.
Hence, a matching partitions the set of agents into pairs and singletons.
- We write $\mu \succ_S \mu'$ if for all $i \in S$, $\mu(i) \succ_i \mu'(i)$.
- Furthermore, $\mu \succ \mu'$ if $\mu \succ_S \mu'$ for some coalition S .
- For a matching μ , S is a *blocking coalition* if there exists a matching μ' such that

$$\mu'(S) = S \text{ and } \mu' \succ_S \mu.$$

- Two special types of blocking coalitions are *single agent coalitions* $\{i\}$ and *blocking pairs* $\{i, j\}$.

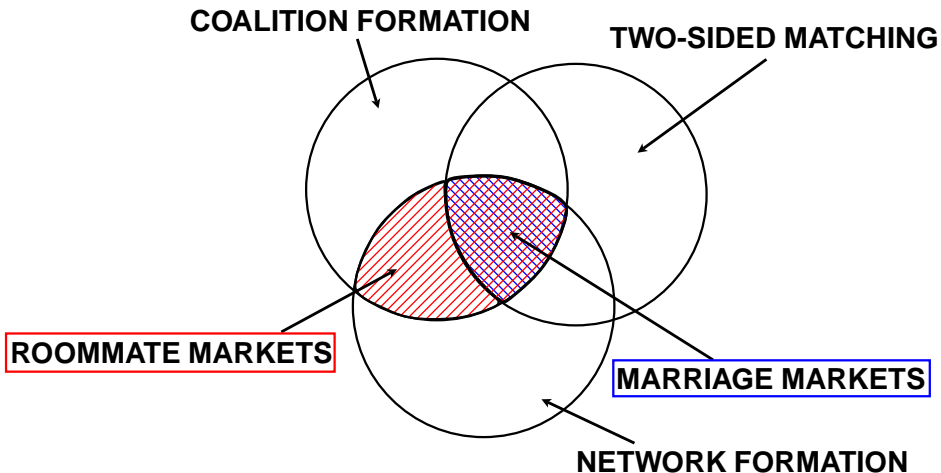
Individual Rationality, Stability, and the Core

- Matching μ is *individually rational* if no single agent blocks μ .
- For **marriage markets**, an individually rational matching never matches two men or two women.
- Matching μ is *stable* if it is *individually rational* and *no blocking pairs exist*.
- Similarly as in other matching models (e.g., marriage markets and college admission markets), the *core*

$$\text{core}(N, \succeq) = \{\mu \mid \text{for all } \mu' \neq \mu, \mu' \not\succeq \mu\}$$

equals the set of stable matchings.

Why Roommate Markets?



The Core for Marriage and Roommate Markets

- For **marriage markets** and **college admission markets** the core is always non-empty and has the very strong structure of a distributive lattice that reflects the polarization between the two sides of the market.

In addition, there is an easy and fast algorithm to find a stable matching (Gale and Shapley's deferred acceptance algorithm).

- For **roommate markets** the core can be empty and if it is non-empty its structure is more complex.

Example I: A Roommate Market with an Empty Core

Example

Agent 1: $2 \succ_1 3 \succ_1 1$,

Agent 2: $3 \succ_2 1 \succ_2 2$,

Agent 3: $1 \succ_3 2 \succ_3 3$.

- All agents being single is not a core matching.
- If agents 1 and 2 are matched, then agent 3 will “seduce” agent 2 to block.
- If agents 2 and 3 are matched, then agent 1 will “seduce” agent 3 to block.
- If agents 1 and 3 are matched, then agent 2 will “seduce” agent 1 to block.

A roommate market with a non-empty core is called *solvable*.

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What if agents are farsighted?

- **Abstract/social situations**

Harsanyi (1974),
Greenberg (1990),
Chwe (1994),
Xue (1998),
Luo (2001, 2006)

- **Hedonic coalition formation**

Diamantoudi and Xue (2003)

- **Network formation**

Page, Wooders, and Kamat (2005),
Herings, Mauleon, and Vannetelbosch (2009),
Page and Wooders (2008)

- **Marriage markets**

Mauleon, Vannetelbosch, and Vergote (2008)

Indirect Dominance (Harsanyi, 1974)

Motivation: A coalition might enforce a myopically not very attractive outcome in order to set a chain of events in motion which in the end will lead to a preferred outcome for the coalition.

Enforceability

- We assume that a coalition $S \subseteq N$ can decide how agents within the coalition are matched.

We therefore consider the following “*enforceability*” notion:

- For any matching μ and any coalition $S \subseteq N$ we say that μ' *results from μ by matching S* if

$$\mu'(S) = S$$

and for all $k \in N \setminus S$,

$$\mu'(k) = \begin{cases} k & \text{if } \mu(k) \in S, \\ \mu(k) & \text{if } \mu(k) \notin S; \end{cases}$$

i.e., coalition S is (re)matched among itself, previous mates of agents in S who are not in S themselves become single, and all other agents have the same mates as before.

- We write this as $\mu \rightarrow_S \mu'$.

Indirect Dominance

Matching μ' *indirectly dominates* matching μ , denoted as $\mu' \gg \mu$, if there exists a chain of matchings

$$\mu = \mu_1 \rightarrow_{\{i_1, j_1\}} \mu_2 \rightarrow_{\{i_2, j_2\}} \cdots \rightarrow_{\{i_{L-1}, j_{L-1}\}} \mu_L = \mu'$$

such that for all $l \in \{1, \dots, L-1\}$,

$$\mu' \succ_{i_l} \mu_l \text{ and } \mu' \succ_{j_l} \mu_l.$$

A Characterization of Indirect Dominance

Proposition

Let $\mu, \mu', \mu \neq \mu'$, be individually rational matchings. Then,
 $\mu' \gg \mu \iff$ **there is no blocking pair $\{i, j\}$ for μ' with $\mu(i) = j$.**
 In particular, if μ' is stable then $\mu' \gg \mu$ for all $\mu \neq \mu'$.

Proof.

" \Rightarrow " Suppose $\mu' \gg \mu$ and there exists a blocking pair $\{i, j\}$ for μ' with $\mu(i) = j$. Hence, there exists a sequence of matchings

$$\mu = \mu_1 \rightarrow_{\{i_1, j_1\}} \mu_2 \rightarrow_{\{i_2, j_2\}} \cdots \rightarrow_{\{i_{L-1}, j_{L-1}\}} \mu_L = \mu'.$$

But, neither i nor j can benefit from participating in such a chain because it ends in μ' where both agents are worse off.

" \Leftarrow " is more complicated. □

von Neumann-Morgenstern Farsightedly Stable Sets

- A **von Neumann-Morgenstern (vNM) farsightedly stable set** is a set of matchings V that satisfies
 - farsighted internal stability** for all $\mu, \mu' \in V$, $\mu' \not\gg \mu$ and
 - farsighted external stability** for all $\mu \notin V$, $\mu' \gg \mu$ for some $\mu' \in V$.
- The vNM farsightedly stable sets represent Greenberg's (1990) *optimistic stable standard of behavior (OSSB)*.
- Mauleon, Vannetelbosch, and Vergote (2008, Core Discussion Paper 2008/16) consider vNM farsightedly stable sets for marriage and college admission markets.

Results

Lemma

*Let V be a vNM farsightedly stable set and $\mu \in V$.
Then, μ is individually rational.*

Theorem

$V = \{\mu\}$ is a vNM farsightedly stable set if and only if μ is stable.

Example I Revisited

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Agent 1: $2 \succ_1 3 \succ_1 1$,

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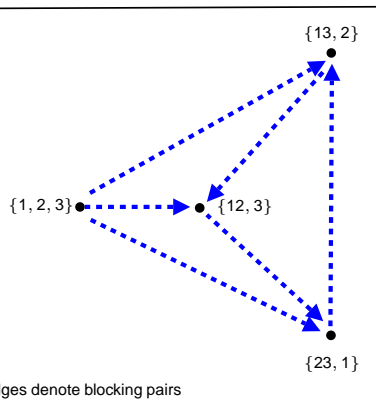
- Let $\mu_1 := \{12, 3\}$, $\mu_2 := \{13, 2\}$, $\mu_3 := \{23, 1\}$, and $\mu_0 := \{1, 2, 3\}$.
- The set of (feasible) matchings is $\{\mu_0, \mu_1, \mu_2, \mu_3\}$.
- Using the characterization of indirect dominance, the *only* indirect dominance relations are
 $\mu_1 \gg \mu_2 \gg \mu_3 \gg \mu_1$ and $\mu_1, \mu_2, \mu_3 \gg \mu_0$.
- Hence, there is no vNM farsightedly stable set.

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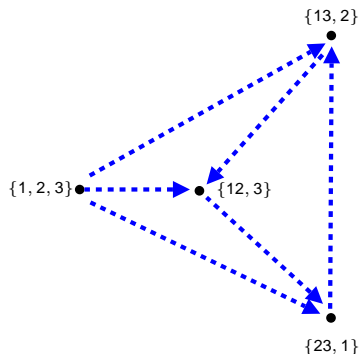


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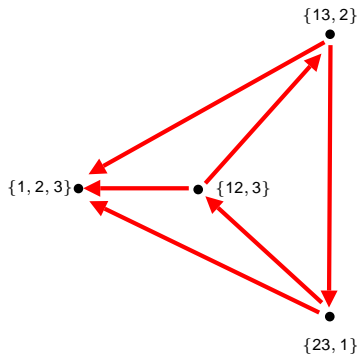
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Edges denote blocking pairs



Edges denote indirect domination

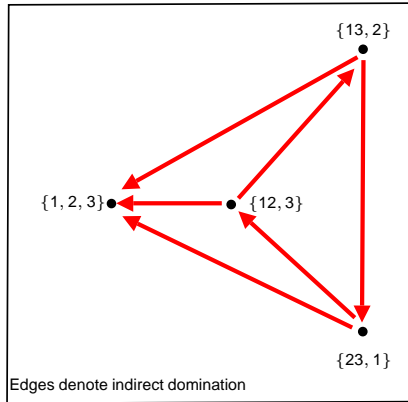
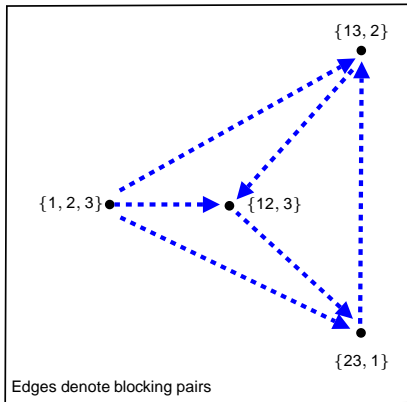
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There is no vNM farsightedly stable set!



Results

Lemma

- *For any vNM farsightedly stable set V of a roommate market, $|V| \neq 2$.*

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- For any *vNM farsightedly stable set* V of a roommate market, $|V| \neq 2$.
- For any *vNM farsightedly stable set* V of a **marriage** market, $|V| \neq 3$.

Results

The results and the example may suggest

- Conjecture: the vNM farsightedly stable sets of any roommate market are the (possibly non-existent) stable singletons.

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- Conjecture: the vNM farsightedly stable sets of any roommate market are the (possibly non-existent) stable singletons.
- **NO!** (See following example.)

Example II: $|V| = 3$ is possible

Example

Agent 1: $2 \succ_1 3 \succ_1 1 \dots$, Agent 4: $6 \succ_4 5 \succ_4 4 \dots$,
 Agent 2: $3 \succ_2 1 \succ_2 2 \dots$, Agent 5: $4 \succ_5 6 \succ_5 5 \dots$,
 Agent 3: $1 \succ_3 2 \succ_3 3 \dots$, Agent 6: $5 \succ_6 4 \succ_6 6 \dots$.

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- Let $\mu_1 := \{12, 3, 45, 6\}$, $\mu_2 := \{23, 1, 56, 4\}$, $\mu_3 := \{13, 2, 46, 5\}$.

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- Let $\mu_1 := \{12, 3, 45, 6\}$, $\mu_2 := \{23, 1, 56, 4\}$, $\mu_3 := \{13, 2, 46, 5\}$.
- $V := \{\mu_1, \mu_2, \mu_3\}$ is farsighted internally stable [characterization of indirect dominance].

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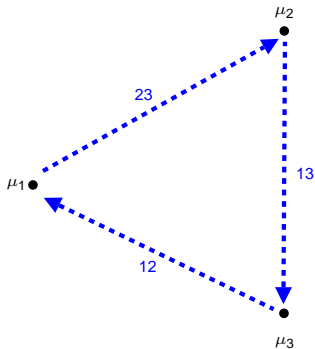
- Let $\mu_1 := \{12, 3, 45, 6\}$, $\mu_2 := \{23, 1, 56, 4\}$, $\mu_3 := \{13, 2, 46, 5\}$.
- $V := \{\mu_1, \mu_2, \mu_3\}$ is farsighted internally stable [characterization of indirect dominance].
- V is farsighted externally stable [tedious construction of indirect dominance paths].

Example II: $|V| = 3$ is possible

$$\mu_1 = \{1, 2, 3\}$$

$$\mu_2 = \{2, 3, 1\}$$

$$\mu_3 = \{1, 3, 2\}$$



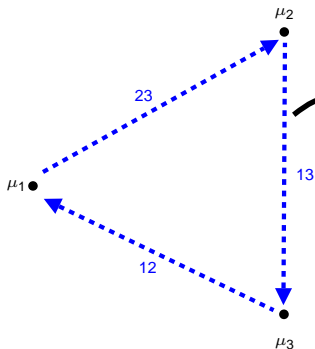
All edges denote blocking pairs

Example II: $|V| = 3$ is possible

$$\mu_1 = \{12, 3 \quad \}$$

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implies $\mu_2 \succcurlyeq \mu_3$

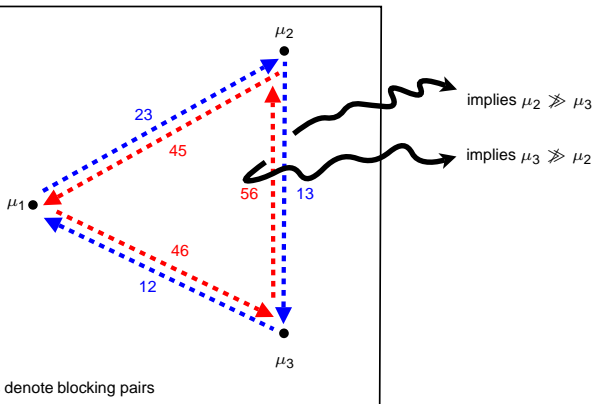
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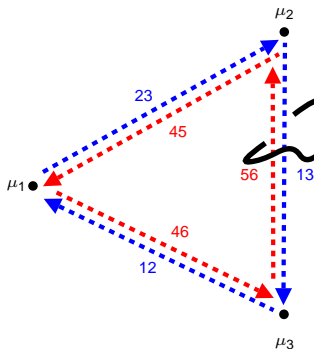
$$\mu_2 = \{23, 1, 56, 4\}$$

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$V = \{\mu_1, \mu_2, \mu_3\}$ is vNM farsightedly internally stable



Similarly, $\mu_1 \succcurlyeq \mu_2, \mu_1 \succcurlyeq \mu_3, \mu_2 \succcurlyeq \mu_1, \mu_3 \succcurlyeq \mu_1$.



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Concluding Remarks

- Two fundamental differences between marriage markets and roommate markets. For a roommate market,
 - there is possibly no vNM farsightedly stable set (Example I),
 - a vNM farsightedly stable set is not necessarily a singleton (Example II).

Concluding Remarks

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 - there is possibly no vNM farsightedly stable set (Example I),
 - a vNM farsightedly stable set is not necessarily a singleton (Example II).
- For a non-solvable roommate market a vNM farsightedly stable set may or may not exist (Examples I and II).

Open Questions

- Structure of vNM farsightedly stable sets?
- Existence of a *solvable* roommate market with a non-singleton vNM farsightedly stable set?