

# Farsighted Stability for Roommate Markets

Bettina Klaus, Flip Klijn, and Markus Walzl

Lausanne, HBS + IAE, Bamberg

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# Roommate Markets

- Gale and Shapley (1962)
- $N = \{1, \dots, n\}$ : *set of agents*.
- $\succsim_i$ : *agent  $i$ 's preferences* over sharing a room with any of the agents in  $N \setminus \{i\}$  and having a room off campus.
- We assume that preferences are *strict*, e.g.,  $j \succ_i k \succ_i i \succ_i h \succ_i \dots$
- A *roommate market* consists of a set of agents  $N$  and their preferences  $\succsim$  and is denoted by  $(N, \succsim)$ .
- A *marriage market* is a roommate market  $(N, \succsim)$  such that  $N$  is the union of two disjoint sets  $M$  and  $W$ , and each agent in  $M$  (respectively  $W$ ) prefers being single to being matched with any other agent in  $M$  (respectively  $W$ ).

# Blocking

- A *matching*  $\mu$  for roommate market  $(N, \succeq)$  is a function  $\mu : N \rightarrow N$  of order two, i.e., for all  $i \in N$ ,  $\mu(\mu(i)) = i$ .  
Hence, a matching partitions the set of agents into pairs and singletons.
- We write  $\mu \succ_S \mu'$  if for all  $i \in S$ ,  $\mu(i) \succ_i \mu'(i)$ .
- Furthermore,  $\mu \succ \mu'$  if  $\mu \succ_S \mu'$  for some coalition  $S$ .
- For a matching  $\mu$ ,  $S$  is a *blocking coalition* if there exists a matching  $\mu'$  such that

$$\mu'(S) = S \text{ and } \mu' \succ_S \mu.$$

- Two special types of blocking coalitions are *single agent coalitions*  $\{i\}$  and *blocking pairs*  $\{i, j\}$ .

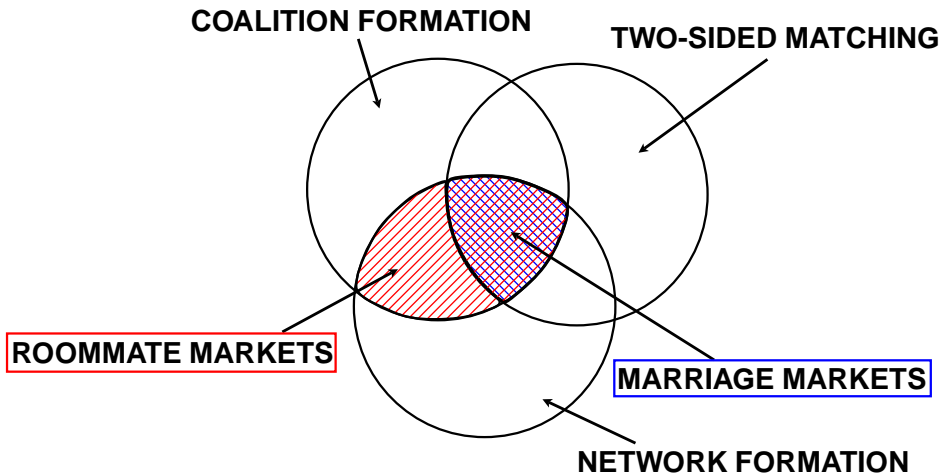
# Individual Rationality, Stability, and the Core

- Matching  $\mu$  is *individually rational* if no single agent blocks  $\mu$ .
- For **marriage markets**, an individually rational matching never matches two men or two women.
- Matching  $\mu$  is *stable* if it is *individually rational* and *no blocking pairs exist*.
- Similarly as in other matching models (e.g., marriage markets and college admission markets), the *core*

$$\text{core}(N, \succeq) = \{\mu \mid \text{for all } \mu' \neq \mu, \mu' \not\succeq \mu\}$$

equals the set of stable matchings.

# Why Roommate Markets?



# The Core for Marriage and Roommate Markets

- For **marriage markets** and **college admission markets** the core is always non-empty and has the very strong structure of a distributive lattice that reflects the polarization between the two sides of the market.

In addition, there is an easy and fast algorithm to find a stable matching (Gale and Shapley's deferred acceptance algorithm).

- For **roommate markets** the core can be empty and if it is non-empty its structure is more complex.

# Example I: A Roommate Market with an Empty Core

## Example

Agent 1:  $2 \succ_1 3 \succ_1 1$ ,

Agent 2:  $3 \succ_2 1 \succ_2 2$ ,

Agent 3:  $1 \succ_3 2 \succ_3 3$ .

- All agents being single is not a core matching.
- If agents 1 and 2 are matched, then agent 3 will “seduce” agent 2 to block.
- If agents 2 and 3 are matched, then agent 1 will “seduce” agent 3 to block.
- If agents 1 and 3 are matched, then agent 2 will “seduce” agent 1 to block.

A roommate market with a non-empty core is called *solvable*.

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# What if agents are farsighted?

- **Abstract/social situations**

Harsanyi (1974),  
Greenberg (1990),  
Chwe (1994),  
Xue (1998),  
Luo (2001, 2006)

- **Hedonic coalition formation**

Diamantoudi and Xue (2003)

- **Network formation**

Page, Wooders, and Kamat (2005),  
Herings, Mauleon, and Vannetelbosch (2009),  
Page and Wooders (2008)

- **Marriage markets**

Mauleon, Vannetelbosch, and Vergote (2008)

# Indirect Dominance (Harsanyi, 1974)

**Motivation:** A coalition might enforce a myopically not very attractive outcome in order to set a chain of events in motion which in the end will lead to a preferred outcome for the coalition.

# Enforceability

- We assume that a coalition  $S \subseteq N$  can decide how agents within the coalition are matched.

We therefore consider the following “*enforceability*” notion:

- For any matching  $\mu$  and any coalition  $S \subseteq N$  we say that  $\mu'$  *results from  $\mu$  by matching  $S$*  if

$$\mu'(S) = S$$

and for all  $k \in N \setminus S$ ,

$$\mu'(k) = \begin{cases} k & \text{if } \mu(k) \in S, \\ \mu(k) & \text{if } \mu(k) \notin S; \end{cases}$$

i.e., coalition  $S$  is (re)matched among itself, previous mates of agents in  $S$  who are not in  $S$  themselves become single, and all other agents have the same mates as before.

- We write this as  $\mu \rightarrow_S \mu'$ .

# Indirect Dominance

Matching  $\mu'$  *indirectly dominates* matching  $\mu$ , denoted as  $\mu' \gg \mu$ , if there exists a chain of matchings

$$\mu = \mu_1 \rightarrow_{\{i_1, j_1\}} \mu_2 \rightarrow_{\{i_2, j_2\}} \cdots \rightarrow_{\{i_{L-1}, j_{L-1}\}} \mu_L = \mu'$$

such that for all  $l \in \{1, \dots, L-1\}$ ,

$$\mu' \succ_{i_l} \mu_l \text{ and } \mu' \succ_{j_l} \mu_l.$$

# A Characterization of Indirect Dominance

## Proposition

Let  $\mu, \mu', \mu \neq \mu'$ , be individually rational matchings. Then,  
 $\mu' \gg \mu \iff$  **there is no blocking pair  $\{i, j\}$  for  $\mu'$  with  $\mu(i) = j$ .**  
 In particular, if  $\mu'$  is stable then  $\mu' \gg \mu$  for all  $\mu \neq \mu'$ .

## Proof.

" $\Rightarrow$ " Suppose  $\mu' \gg \mu$  and there exists a blocking pair  $\{i, j\}$  for  $\mu'$  with  $\mu(i) = j$ . Hence, there exists a sequence of matchings

$$\mu = \mu_1 \rightarrow_{\{i_1, j_1\}} \mu_2 \rightarrow_{\{i_2, j_2\}} \cdots \rightarrow_{\{i_{L-1}, j_{L-1}\}} \mu_L = \mu'.$$

But, neither  $i$  nor  $j$  can benefit from participating in such a chain because it ends in  $\mu'$  where both agents are worse off.

" $\Leftarrow$ " is more complicated. □

# von Neumann-Morgenstern Farsightedly Stable Sets

- A **von Neumann-Morgenstern (vNM) farsightedly stable set** is a set of matchings  $V$  that satisfies
  - farsighted internal stability** for all  $\mu, \mu' \in V$ ,  $\mu' \not\gg \mu$  and
  - farsighted external stability** for all  $\mu \notin V$ ,  $\mu' \gg \mu$  for some  $\mu' \in V$ .
- The vNM farsightedly stable sets represent Greenberg's (1990) *optimistic stable standard of behavior (OSSB)*.
- Mauleon, Vannetelbosch, and Vergote (2008, Core Discussion Paper 2008/16) consider vNM farsightedly stable sets for marriage and college admission markets.

# Results

## Lemma

*Let  $V$  be a vNM farsightedly stable set and  $\mu \in V$ .  
Then,  $\mu$  is individually rational.*

## Theorem

*$V = \{\mu\}$  is a vNM farsightedly stable set if and only if  $\mu$  is stable.*

# Example I Revisited

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Agent 1:  $2 \succ_1 3 \succ_1 1$ ,

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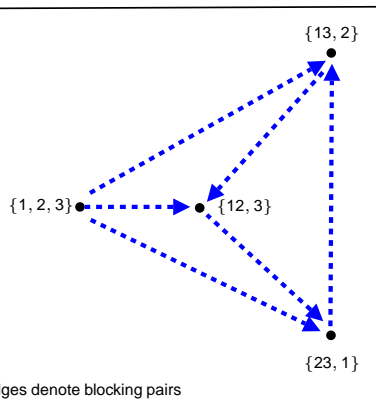
- Let  $\mu_1 := \{12, 3\}$ ,  $\mu_2 := \{13, 2\}$ ,  $\mu_3 := \{23, 1\}$ , and  $\mu_0 := \{1, 2, 3\}$ .
- The set of (feasible) matchings is  $\{\mu_0, \mu_1, \mu_2, \mu_3\}$ .
- Using the characterization of indirect dominance, the *only* indirect dominance relations are  
 $\mu_1 \gg \mu_2 \gg \mu_3 \gg \mu_1$  and  $\mu_1, \mu_2, \mu_3 \gg \mu_0$ .
- Hence, there is no vNM farsightedly stable set.

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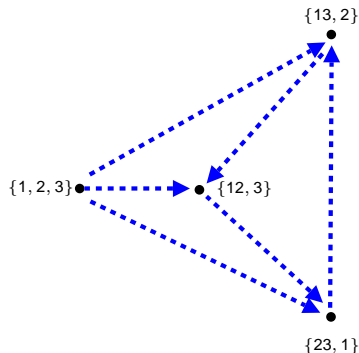


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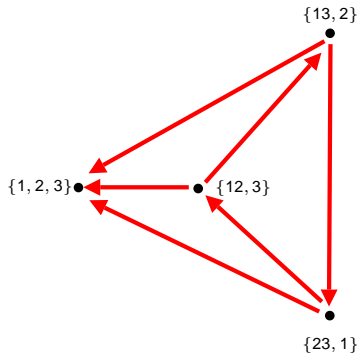
Agent 1:  $2 \succ_1 3 \succ_1 1$

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Edges denote blocking pairs



Edges denote indirect domination

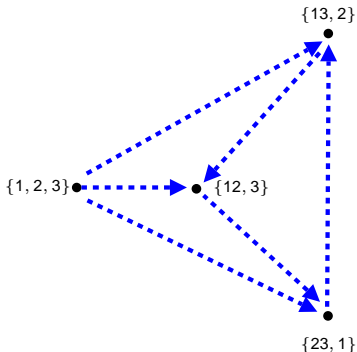
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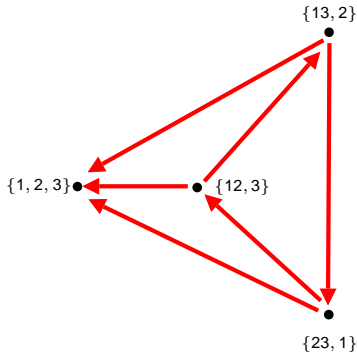
Agent 2:  $3 \succ_2 1 \succ_2 2$

Agent 3:  $1 \succ_3 2 \succ_3 3$

There is no vNM farsightedly stable set!



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# Results

## Lemma

- *For any vNM farsightedly stable set  $V$  of a roommate market,  $|V| \neq 2$ .*

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## Lemma

- For any vNM farsightedly stable set  $V$  of a roommate market,  $|V| \neq 2$ .
- For any vNM farsightedly stable set  $V$  of a **marriage** market,  $|V| \neq 3$ .

# Results

The results and the example may suggest

- Conjecture: the vNM farsightedly stable sets of any roommate market are the (possibly non-existent) stable singletons.

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- Conjecture: the vNM farsightedly stable sets of any roommate market are the (possibly non-existent) stable singletons.
- **NO!** (See following example.)

# Example II: $|V| = 3$ is possible

## Example

Agent 1:  $2 \succ_1 3 \succ_1 1 \dots$ , Agent 4:  $6 \succ_4 5 \succ_4 4 \dots$ ,  
 Agent 2:  $3 \succ_2 1 \succ_2 2 \dots$ , Agent 5:  $4 \succ_5 6 \succ_5 5 \dots$ ,  
 Agent 3:  $1 \succ_3 2 \succ_3 3 \dots$ , Agent 6:  $5 \succ_6 4 \succ_6 6 \dots$ .

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- Let  $\mu_1 := \{12, 3, 45, 6\}$ ,  $\mu_2 := \{23, 1, 56, 4\}$ ,  $\mu_3 := \{13, 2, 46, 5\}$ .

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- $V := \{\mu_1, \mu_2, \mu_3\}$  is farsighted internally stable [characterization of indirect dominance].

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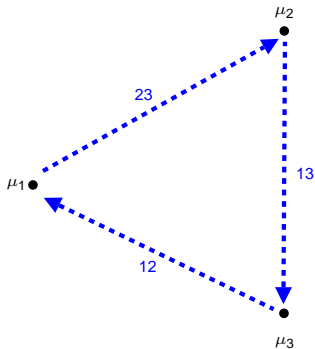
- Let  $\mu_1 := \{12, 3, 45, 6\}$ ,  $\mu_2 := \{23, 1, 56, 4\}$ ,  $\mu_3 := \{13, 2, 46, 5\}$ .
- $V := \{\mu_1, \mu_2, \mu_3\}$  is farsighted internally stable [characterization of indirect dominance].
- $V$  is farsighted externally stable [tedious construction of indirect dominance paths].

# Example II: $|V| = 3$ is possible

$$\mu_1 = \{1, 2, 3\}$$

$$\mu_2 = \{2, 3, 1\}$$

$$\mu_3 = \{1, 3, 2\}$$



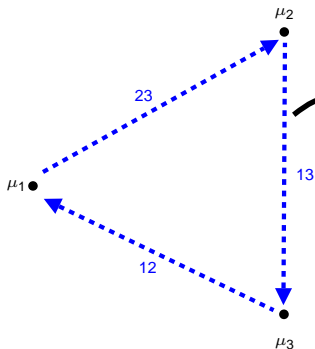
All edges denote blocking pairs

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$$\mu_1 = \{1, 2, 3\}$$

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implies  $\mu_2 \succcurlyeq \mu_3$

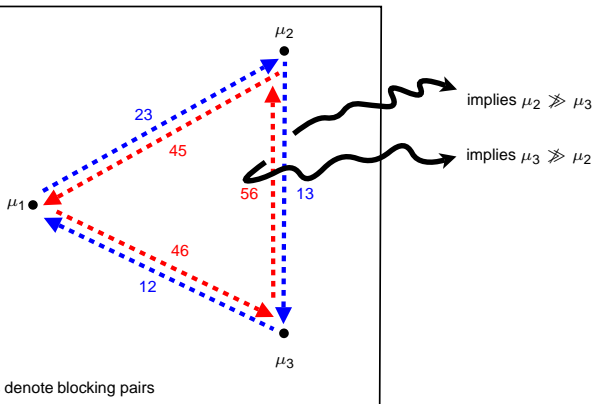
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Example II:  $|V| = 3$  is possible

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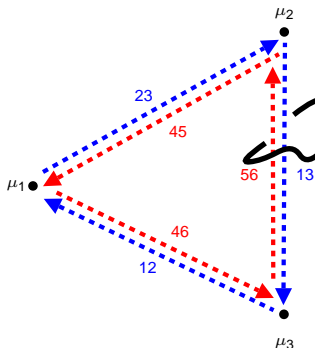
$$\mu_2 = \{23, 1, 56, 4\}$$

$$\mu_3 = \{13, 2, 46, 5\}$$

$V = \{\mu_1, \mu_2, \mu_3\}$  is vNM farsightedly internally stable



Similarly,  $\mu_1 \succcurlyeq \mu_2, \mu_1 \succcurlyeq \mu_3, \mu_2 \succcurlyeq \mu_1, \mu_3 \succcurlyeq \mu_1$ .



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# Concluding Remarks

- Two fundamental differences between marriage markets and roommate markets. For a roommate market,
  - there is possibly no vNM farsightedly stable set (Example I),
  - a vNM farsightedly stable set is not necessarily a singleton (Example II).

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  - there is possibly no vNM farsightedly stable set (Example I),
  - a vNM farsightedly stable set is not necessarily a singleton (Example II).
- For a non-solvable roommate market a vNM farsightedly stable set may or may not exist (Examples I and II).

# Open Questions

- Structure of vNM farsightedly stable sets?
- Existence of a *solvable* roommate market with a non-singleton vNM farsightedly stable set?