Farsighted Housing Allocation

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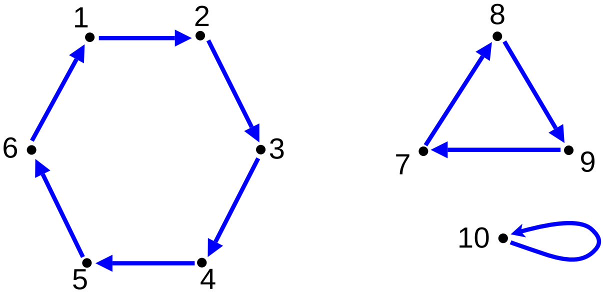
Housing Markets

- Shapley and Scarf (1974)
- A fixed *set of agents* $N = \{1, \ldots, n\}$.
- The *endowment* of agent i is one object denoted by $e_i = i$.
- R_i denotes agent i's complete and transitive preferences over objects. (Usual notation P_i and I_i.)
- A *housing market* is a profile of preferences $(R_i)_{i \in N}$.
- An *allocation* is an assignment of objects $x = (x_i)_{i \in N}$ such that for each $i \in N$, agent i receives exactly one object, denoted by x_i.
- An allocation x can be represented by a *directed graph g* with nodes N and a directed edge from agent i to agent j if $x_i = j$.
- The graph g partitions the set of agents N into *trading cycles* C^l $(l = 1, ..., L)$.

Trading Cycles

Let $N = \{1, \ldots, 10\}$.

For allocation $x = (2, 3, 4, 5, 6, 1, 8, 9, 7, 10)$ the graph g is as follows:



There are 3 trading cycles: (1,2,3,4,5,6), (7,8,9), and (10).

Why Housing Markets?

Recent applications:

- Allocation of student housing with existing tenants (Abdulkadiroğlu and Sönmez, 1999).
- Kidney exchange (Roth, Sönmez, and Ünver, 2004).

Prominent Solution Concepts

- comp := competitive allocations
- TTC := outcomes of Top Trading Cycles algorithm (Gale)
- SC := strong core (i.e., based on weak domination)

comp = TTC (Shapley and Scarf, 1974);
comp = SC is a singleton (Roth and Postlewaite, 1977) for *strict* pref.

When preferences are *not strict*: possibly SC $= \emptyset$, and

- WC := weak core (i.e., based on strong domination)

is non-empty but possibly comp $\subsetneq$ WC.

Wako (1999) proposes a core based on antisymmetric weak domination and shows that it coincides with comp.
Toda (1997) shows that the unique vNM stable set based on antisymmetric weak domination coincides with comp.

Indirect Dominance

Motivation: A coalition might enforce a myopically not very attractive outcome in order to set a chain of events in motion which in the end will lead to a preferred outcome for the coalition (Harsanyi, 1974).

Kawasaki (2008) demonstrates that comp coincides with the unique von Neumann-Morgenstern stable set defined in terms of a farsighted version of Wako's (1999) antisymmetric weak dominance.

In this paper, we give a sufficient 'reasonable' preference domain on which comp coincides with the unique von Neumann-Morgenstern stable set defined in terms of a farsighted version of strong dominance.

Enforceability

For an allocation x, let $C^{x,k}$ denote the trading cycle containing k.

A coalition S of agents can *enforce* an allocation y starting from an allocation x, denoted by $x \rightarrow_S y$, if

(E1) $\cup_{i \in S}\{y_i\} = S$;

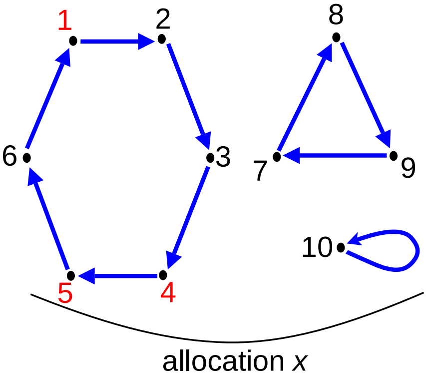
(E2) $y_k = x_k$ for all $k \in N \backslash S$ with $S \cap C^{x,k} = \emptyset$;

(E3) $y_k = e_k$ for all $k \in N \backslash S$ with $S \cap C^{x,k} \neq \emptyset$.

Enforceability

Let $N = \{1, \ldots, 10\}$.

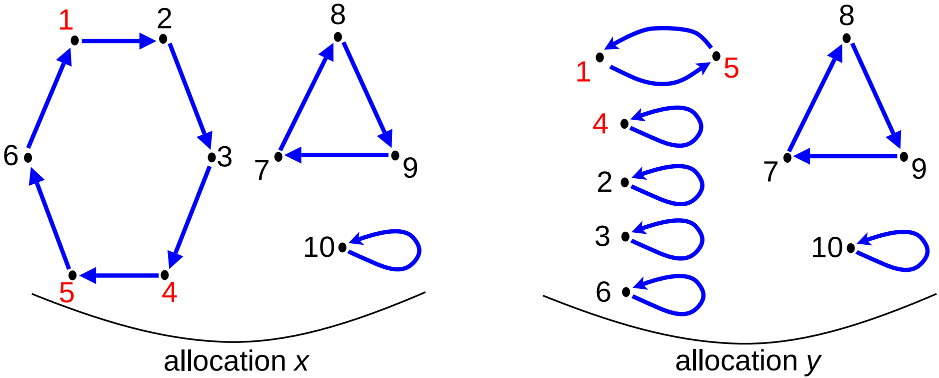
Starting from allocation $x = (2, 3, 4, 5, 6, 1, 8, 9, 7, 10)$,



Enforceability

Let $N = \{1, \ldots, 10\}$.

Starting from allocation $x = (2, 3, 4, 5, 6, 1, 8, 9, 7, 10)$,
coalition $\{1, 4, 5\}$ can enforce allocation $y = (5, 2, 3, 4, 1, 6, 8, 9, 7, 10)$.



Indirect Dominance

Allocation y *indirectly dominates* allocation x, denoted as $y \gg x$, if there exists a chain of allocations

$$x = x^1 \longrightarrow_{S^1} x^2 \longrightarrow_{S^2} \cdots \longrightarrow_{S^{L-1}} x^L = y$$

such that for all $l \in \{1, \ldots, L-1\}$ and for all $k \in S^l$,

$$y_k \, P_k \, x_k^l.$$

von Neumann-Morgenstern Farsightedly Stable Sets

A von Neumann-Morgenstern (vNM) farsightedly stable set is a set of allocations V that satisfies

farsighted internal stability for all $x, y \in V$, $y \not\gg x$ and

farsighted external stability for all $x \notin V$, $y \gg x$ for some $y \in V$.

The vNM farsightedly stable sets represent Greenberg's (1990) *optimistic stable standard of behavior* (OSSB).

Results

Lemma

For any housing market, comp is vNM farsightedly internally stable.

Proof.

Let $x \in$ comp = TTC. Consider (from top to bottom) the 'layers' of agents in the Top Trading Cycles algorithm.

One proves, layer by layer, that no trading cycle agent can form part of an indirect dominance path that would start in x.

This shows that x is not indirectly dominated by *any* other allocation (not even by those outside comp). □

Results

- Question: For any housing market, is comp also vNM farsightedly **ex**ternally stable?

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- Question: For any housing market, is comp also vNM farsightedly **ex**ternally stable?
- **NO!** (See following example.)

Example: comp is not farsightedly externally stable

Example

Agent 1: $\bar{2}\ P_1\ 3\ P_1\ \underline{1}$,
Agent 2: $\bar{3}\ P_2\ \underline{2}\ P_2\ 1$,
Agent 3: $\bar{1}\ I_3\ \underline{3}\ P_3\ 2$.

- Let $y^1 := (2, 3, 1)$ and $y^2 := (1, 2, 3)$. Then, comp = $\{y^1, y^2\}$.

Example: comp is not farsightedly externally stable

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- Let $y^1 := (2,3,1)$ and $y^2 := (1,2,3)$. Then, comp = $\{y^1, y^2\}$.
- Consider $x := (3,2,1)$. Note: for all i, $x_i\ R_i\ y_i^2$. So, $y^2 \not\gg x$.

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- Let $y^1 := (2,3,1)$ and $y^2 := (1,2,3)$. Then, comp = $\{y^1, y^2\}$.
- Consider $x := (3,2,1)$. Note: for all i, $x_i\ R_i\ y_i^2$. So, $y^2 \not\gg x$.
- Suppose $y^1 \gg x$. Since $x_3 = y_3^1$, $S_1 \subseteq \{1,2\}$. W.l.o.g. $x^2 \neq x^1$. By (E2), $S^1 \neq \{2\}$. So, $S_1 = \{1\}$ or $S_1 = \{1,2\}$.

Example: comp is not farsightedly externally stable

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- Let $y^1 := (2,3,1)$ and $y^2 := (1,2,3)$. Then, comp = $\{y^1, y^2\}$.
- Consider $x := (3,2,1)$. Note: for all i, $x_i \, R_i \, y_i^2$. So, $y^2 \not\gg x$.
- Suppose $y^1 \gg x$. Since $x_3 = y_3^1$, $S_1 \subseteq \{1,2\}$. W.l.o.g. $x^2 \neq x^1$. By (E2), $S^1 \neq \{2\}$. So, $S_1 = \{1\}$ or $S_1 = \{1,2\}$.
- By (E1,E2), $x^2 = (1,2,3)$ or $x^2 = (2,1,3)$. In either case, $x_3^2 = 3$.

Example: comp is not farsightedly externally stable

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$$\text{Agent 1: } \bar{2}\ P_1\ 3\ P_1\ \underline{1},$$
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- Let $y^1 := (2,3,1)$ and $y^2 := (1,2,3)$. Then, comp = $\{y^1, y^2\}$.
- Consider $x := (3,2,1)$. Note: for all i, $x_i\, R_i\, y_i^2$. So, $y^2 \not\gg x$.
- Suppose $y^1 \gg x$. Since $x_3 = y_3^1$, $S_1 \subseteq \{1,2\}$. W.l.o.g. $x^2 \neq x^1$. By (E2), $S^1 \neq \{2\}$. So, $S_1 = \{1\}$ or $S_1 = \{1,2\}$.
- By (E1,E2), $x^2 = (1,2,3)$ or $x^2 = (2,1,3)$. In either case, $x_3^2 = 3$.
- Since for all j, $3\, R_3\, j$, we have for all $l = 2, \ldots, L-1$, $3 \notin S^l$. In particular, $3 = x_3^2 = x_3^3 = \ldots = x_3^L = y_3^1 = 1$; a contradiction. Hence, $y^1 \not\gg x$.

Restricted Domain

Observation: What hinders farsighted external stability of comp in the Example is the fact that agent 3 is ***indifferent between his own endowment and some other object***.
(Indifferences not of this type would not be 'problematic.')

However, a preference restriction of this type seems to be realistic for standard applications of housing market models.

For instance, it is reasonable to assume that a patient/donor pair strictly prefers its own donor kidney to any kidney with the same medical characteristics, while it also strictly prefers any kidney with better medical characteristics to its own.

Likewise, a tenant strictly prefers his own room to any room with identical characteristics —he likes to avoid moving—, while he also strictly prefers any room with distinctly better characteristics to his own.

Restricted Domain

The Example shows that

- if one agent is indifferent between his endowment and the endowment of one other agent
- then the set of competitive allocations need not be vNM farsightedly externally stable.

Restricted Domain: Results

Lemma

For any housing market in 'the restricted domain', comp is vNM farsightedly externally stable.

Proof.

For any $x \in$ comp we construct an indirect dominance path to some competitive allocation.

The construction is based on the alternative application of two procedures. If you're curious (but of course you are)... read the paper! □

Restricted Domain: Results

Theorem

*For any housing market in 'the restricted domain', comp is the **unique** vNM farsightedly stable set.*

Proof.

From the Lemmas if follows immediately that comp is a vNM farsightedly stable set.

The uniqueness follows from the slightly stronger statement shown in the proof of the first Lemma. □